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# On index-2 linear implicit difference equations ☆

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ABSTRACT

This paper deals with an index-2 notion for linear implicit difference equations (LIDEs) and with the solvability of initial value problems (IVPs) for index-2 LIDEs. Besides, the cocycle property as well as the multiplicative ergodic theorem of Oseledets type are also proved.


## 1. Introduction

Linear implicit difference equations

$$A_n x_{n+1} = B_n x_n + q_n, \quad n \geqslant 0 \tag{1.1}$$

with $A_n, B_n \in \mathbb{R}^{m\times m}$, $q_n \in \mathbb{R}^m$ to be given, arise in many real problems, such as in the Leslie population growth model or in the Leontief dynamic model of multisector economy [3] and so forth. In addition, they can be considered as the discretization of linear differential algebraic equations (DAEs) by explicit Euler method

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$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in J := [t_0, T], \tag{1.2}$$

where $A, B \in C(J, \mathbb{R}^{m\times m}),\ q \in C(J, \mathbb{R}^m)$ and the matrix $A(t)$ is singular for every $t \in J$ (cf. [1]).

For DAEs, in [6,8] a unified approach to Eq. (1.2) has been proposed. By introducing index-1 and index-2 concepts for Eq. (1.2), some solution representations of IVPs for index-1 and index-2 linear DAEs are given. These representations are based on the solutions of a certain inherent regular ordinary differential equations which are uniquely determined by the problem data (see [5,6,8]).

In the case of the LIDE (1.1), a class, called index-1 LIDEs, has been investigated. In [7], the solvability of IVPs for index-1 LIDEs has been studied. A connection between index-1 linear DAEs and index-1 LIDEs has been claimed in [1]. In particular, the compatibility between index-1 notions for linear DAEs and LIDEs has been established. For random LIDEs, the random cocycle property of the solutions as well as the MET theorem for Lyapunov spectrum has been introduced (cf. [4]) with the assumption that these systems are index-1 tractable.

This paper continues studying the LIDE (1.1). We are going to deal with a higher index of LIDEs and apply this concept to study implicit random dynamic systems. Since a higher index definition is based on a lower one, the most difficulty is to show that this definition does not depend on the previous steps. By means of index-2 concept, we can split Eq. (1.1) into an ordinary difference equation and some algebraic relations. Hence, we can introduce a formula for the solution of (1.1) under certain initial conditions. As usual, with these results of index-2 LIDEs we aim at a cocycle property and a decomposition of Lyapunov exponents for the solutions of real noise index-2 LIDEs. In the case of index-1, similar results have been proved in [4].

The paper is organized as follows. In Section 2, the index-1 LIDEs are shortly mentioned. Section 3 deals with an index-2 concept of the LIDEs. In Section 4, we set out the condition of solvability and give an explicit form of solution of IVPs for index-2 LIDEs. Section 5 is concerned with the cocycle property of solution over random dynamics and the multiplicative ergodic theorem MET for the case of index-2 LIDEs. The last section illustrates our results by means of examples. For the convenience of the reader, a short appendix provides some basic linear algebra facts once more.

## 2. Implicit difference equations with index-1 tractable

We now turn to Eq. (1.1). Throughout of this paper, we assume that $\operatorname{rank} A_n \equiv r\ (1 \leqslant r \leqslant m-1)$ for all $n \geqslant 0$. Denote $N_n := \ker A_n$ and let $Q_n$ be a projection onto $N_n$. Put $P_n := I - Q_n$. Let $T_n \in \mathrm{GL}(\mathbb{R}^m)$, $n \geqslant 1$ such that $T_n|_{N_n}$ is an isomorphism from $N_n$ onto $N_{n-1}$. For definiteness, we put $A_{-1} := A_0$. Hence, the operators $T_n$ are determined for all $n \geqslant 0$. Associating to Eq. (1.1) we introduce the following notations

$$G_n := A_n + B_nT_nQ_n, \quad S_n := \{z \in \mathbb{R}^m : B_nz \in \operatorname{im} A_n\}, \quad n \geqslant 0.$$

Using Lemma 1 in Appendix A with $A = A_n$, $\bar{A} = A_{n-1}$ and $B = B_n$ we have the following lemma.

**Lemma 2.1.** *The following conditions are equivalent*

(i) *the matrix* $G_n := A_n + B_nT_nQ_n$ *is nonsingular*;
(ii) $N_{n-1} \oplus S_n = \mathbb{R}^m$;
(iii) $N_{n-1} \cap S_n = \{0\}$.

By virtue of Lemma 2.1 we can define the so-called index-1 tractable LIDEs.

**Definition 2.2.** The LIDE (1.1) is said to be of index-1 tractable (index-1 for short) if for all $n \geqslant 0$, the following conditions

(i) $\operatorname{rank} A_n = r = \text{const}$,
(ii) $N_{n-1} \cap S_n = \{0\}$

hold.

Now, we describe briefly the decomposition technique for index-1 LIDEs. Using Lemma 2.1, we see that the matrices $G_n$ are nonsingular for all $n \geqslant 0$. Hence, multiplying (1.1) by $P_nG_n^{-1}$ and $Q_nG_n^{-1}$, respectively, and applying the formula (A.1) of Lemma 2 in Appendix A with $A = A_n$, $\overline{A} = A_{n-1}$ and $B = B_n$, we decouple Eq. (1.1) into the system

$$P_nx_{n+1} = P_nG_n^{-1}B_nx_n + P_nG_n^{-1}q_n, \tag{2.1}$$
$$0 = Q_nG_n^{-1}B_nx_n + Q_nG_n^{-1}q_n. \tag{2.2}$$

Noting that $P_nT_n^{-1}Q_{n-1} = 0$, $Q_nT_n^{-1}Q_{n-1} = T_n^{-1}Q_{n-1}$ and using Eq. (A.3) we obtain

$$P_nG_n^{-1}B_nx_n = P_n(G_n^{-1}B_nP_{n-1} + T_n^{-1}Q_{n-1})x_n = P_nG_n^{-1}B_nP_{n-1}x_n,$$

and

$$Q_nG_n^{-1}B_nx_n = Q_n(G_n^{-1}B_nP_{n-1} + T_n^{-1}Q_{n-1})x_n = Q_nG_n^{-1}B_nP_{n-1}x_n + T_n^{-1}Q_{n-1}x_n.$$

Therefore, by denoting $u_n := P_{n-1}x_n$, $v_n := Q_{n-1}x_n$ for $n \geqslant 0$, Eqs. (2.1) and (2.2) deduce to

$$\begin{aligned} u_{n+1} &= P_nG_n^{-1}B_nu_n + P_nG_n^{-1}q_n, \\ v_n &= -T_nQ_nG_n^{-1}B_nu_n - T_nQ_nG_n^{-1}q_n. \end{aligned} \tag{2.3}$$

Hence,

$$x_n = u_n + v_n = (I - T_nQ_nG_n^{-1}B_n)u_n - T_nQ_nG_n^{-1}q_n. \tag{2.4}$$

Thus, solving the ordinary difference equation (2.3) with the initial condition $u_0 \in \operatorname{im} P_{-1}$ and using Eq. (2.4), we get an expression of the solution of the index-1 LIDE (1.1).

Inspired by the above decoupling procedure, we state initial conditions for the index-1 LIDE (1.1) as

$$P_{-1}(x_0 - x^0) = 0, \quad x^0 \in \mathbb{R}^m \text{ given.}$$

It yields that

$$u_0 = P_{-1}x_0 = P_{-1}x^0 =: u^0,$$

but we do not expect

$$x_0 = x^0$$

as in the ordinary difference equations. The details can be referred to [4].

## 3. Implicit difference equations with index-2 tractable

In developing further study of LIDEs, we are going to deal with the so-called index-2 concept which is based on the main idea that it is the “index-1 of the index-1". Henceforth, we suppose that the matrices $G_n$ are singular for all $n \geqslant 0$. Denote

$$N_{1,n} := \ker G_n, \quad S_{1,n} := \{z \in \mathbb{R}^m : B_nP_{n-1}z \in \operatorname{im} G_n\}.$$

**Lemma 3.1.** *There holds the following relation*

$$(T_n + T_nP_nA_n^+B_nT_nQ_n)N_{1,n} = N_{n-1} \cap S_n,$$

*where $A_n^+$ is the Moore–Penrose generalized inverse of $A_n$.*

**Proof.** First, we prove the following inclusion

$$(T_n + T_nP_nA_n^+B_nT_nQ_n)N_{1,n} \subseteq N_{n-1} \cap S_n. \tag{3.1}$$

Indeed, let

$$\bar{x} = (T_n + T_nP_nA_n^+B_nT_nQ_n)x \in (T_n + T_nP_nA_n^+B_nT_nQ_n)N_{1,n},$$

where $x \in N_{1,n}$. Since $x \in N_{1,n}$, it implies that $B_nT_nQ_nx = -A_nx$. Therefore,

$$A_{n-1}\bar{x} = A_{n-1}T_n(I - P_nA_n^+A_n)x.$$

By noting that $I - A_n^+A_n$ is a projection onto $N_n$, we get $P_nA_n^+A_n = P_n$. Hence,

$$A_{n-1}\bar{x} = A_{n-1}T_n(I - P_n)x = A_{n-1}T_nQ_n = 0,$$

i.e., $\bar{x} \in N_{n-1}$. Besides,

$$B_n\bar{x} = B_n(T_n + T_nP_nA_n^+B_nT_nQ_n)x = B_nT_n(I - P_n)x = B_nT_nQ_nx = -A_nx \in \operatorname{im} A_n.$$

This says that $\bar{x} \in S_n$. Thus, $\bar{x} \in N_{n-1} \cap S_n$, or the inclusion (3.1) is proved.

Conversely, let $\bar{x} \in N_{n-1} \cap S_n$ be arbitrary. To prove

$$\bar{x} \in (T_n + T_nP_nA_n^+B_nT_nQ_n)N_{1,n},$$

we consider $G_n(T_n + T_nP_nA_n^+B_nT_nQ_n)^{-1}\bar{x}$. Firstly, since $P_n$ and $Q_n$ are projector matrices with $P_n + Q_n = I$, it implies easily that

$$(I + P_nA_n^+B_nT_nQ_n)(I - P_nA_n^+B_nT_nQ_n) = I.$$

Therefore, we have

$$\begin{aligned} G_n(T_n + T_nP_nA_n^+B_nT_nQ_n)^{-1}\bar{x} &= G_n(I - P_nA_n^+B_nT_nQ_n)T_n^{-1}\bar{x} \\ &= (G_n - G_nP_nA_n^+B_nT_nQ_n)T_n^{-1}\bar{x} \\ &= (A_n + B_nT_nQ_n - A_nA_n^+B_nT_nQ_n)T_n^{-1}\bar{x}. \end{aligned}$$

Due to $\bar{x} \in N_{n-1}$, we get $T_n^{-1}\bar{x} \in N_n$, i.e., $A_nT_n^{-1}\bar{x} = 0$ and $Q_nT_n^{-1}\bar{x} = T_n^{-1}\bar{x}$. These relations lead to

$$G_n(T_n + T_nP_nA_n^+B_nT_nQ_n)^{-1}\bar{x} = (I - A_nA_n^+)B_n\bar{x}.$$

Further, $\bar{x} \in S_n$, i.e., there exists $z \in \mathbb{R}^m$ such that $B_n\bar{x} = A_nz$. Hence,

$$G_n(T_n + T_nP_nA_n^+B_nT_nQ_n)^{-1}\bar{x} = (I - A_nA_n^+)A_nz = (A_n - A_nA_n^+A_n)z = (A_n - A_n)z = 0,$$

which implies

$$\bar{x} \in (T_n + T_nP_nA_n^+B_nT_nQ_n)N_{1,n}.$$

Thus, we have already proved the inclusion

$$N_{n-1} \cap S_n \subseteq (T_n + T_nP_nA_n^+B_nT_nQ_n)N_{1,n}. \tag{3.2}$$

By combining the inclusions (3.1) and (3.2), we obtain

$$(T_n + T_nP_nA_n^+B_nT_nQ_n)N_{1,n} = N_{n-1} \cap S_n.$$

Lemma 3.1 is proved. □

As a direct consequence of Lemma 3.1, we get the following corollary:

**Corollary 3.2.** $\dim N_{1,n} = \dim(N_{n-1} \cap S_n)$.

**Lemma 3.3.** *Let $Q_n$ and $\widetilde{Q}_n$ be two projections onto $N_n$, and let $T_n, \widetilde{T}_n \in \mathrm{GL}(\mathbb{R}^m)$ such that $T_n|_{N_n}$, $\widetilde{T}_n|_{N_n}$ are two isomorphisms between $N_n$ and $N_{n-1}$. Put $G_n := A_n + B_nT_nQ_n$, $\widetilde{G}_n := A_n + B_n\widetilde{T}_n\widetilde{Q}_n$ and*

$$\widetilde{N}_{1,n} := \ker \widetilde{G}_n, \quad \widetilde{S}_{1,n} := \{z \in \mathbb{R}^m : B_n\widetilde{P}_{n-1}z \in \operatorname{im} \widetilde{G}_n\}.$$

*Then, there hold the following relations*

(i) $\widetilde{N}_{1,n} = (\widetilde{P}_n + \widetilde{T}_n^{-1}T_nQ_n)N_{1,n}$; (3.3)

(ii) $S_{1,n} = \widetilde{S}_{1,n}$, *i.e.*, $S_{1,n}$ *is independent of the choice of* $Q_n$ *and* $T_n$;
(iii) $S_{1,n} = (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}) S_{1,n}$,

*this means that* $S_{1,n}$ *is an invariant subspace of* $\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}$.

**Proof.** (i) By definition we have

$$\widetilde{G}_n = A_n + B_n \widetilde{T}_n \widetilde{Q}_n = G_n P_n + B_n T_n T_n^{-1} \widetilde{T}_n \widetilde{Q}_n.$$

It is clear that $T_n^{-1} \widetilde{T}_n \widetilde{Q}_n x \in N_n$ for all $x \in \mathbb{R}^m$. Therefore, $Q_n T_n^{-1} \widetilde{T}_n \widetilde{Q}_n x = T_n^{-1} \widetilde{T}_n \widetilde{Q}_n x$ for all $x \in \mathbb{R}^m$. Hence, $Q_n T_n^{-1} \widetilde{T}_n \widetilde{Q}_n = T_n^{-1} \widetilde{T}_n \widetilde{Q}_n$ which implies that

$$\begin{aligned} \widetilde{G}_n &= G_n P_n + B_n T_n Q_n T_n^{-1} \widetilde{T}_n \widetilde{Q}_n = G_n P_n + B_n T_n Q_n Q_n T_n^{-1} \widetilde{T}_n \widetilde{Q}_n \\ &= G_n P_n + (A_n + B_n T_n Q_n) Q_n T_n^{-1} \widetilde{T}_n \widetilde{Q}_n = G_n P_n + G_n T_n^{-1} \widetilde{T}_n \widetilde{Q}_n = G_n (P_n + T_n^{-1} \widetilde{T}_n \widetilde{Q}_n). \end{aligned}$$

Thus, we go to the following equality

$$\widetilde{G}_n = G_n (P_n + T_n^{-1} \widetilde{T}_n \widetilde{Q}_n). \tag{3.4}$$

Moreover, from the relation

$$\begin{aligned} (P_n + T_n^{-1} \widetilde{T}_n \widetilde{Q}_n)(\widetilde{P}_n + \widetilde{T}_n^{-1} T_n Q_n) &= P_n \widetilde{P}_n + P_n \widetilde{T}_n^{-1} T_n Q_n + T_n^{-1} \widetilde{T}_n \widetilde{Q}_n \widetilde{T}_n^{-1} T_n Q_n \\ &= P_n + P_n Q_n \widetilde{T}_n^{-1} T_n Q_n + T_n^{-1} \widetilde{T}_n \widetilde{T}_n^{-1} T_n Q_n \\ &= P_n + Q_n = I, \end{aligned}$$

it follows that $P_n + T_n^{-1} \widetilde{T}_n \widetilde{Q}_n$ is invertible and

$$(P_n + T_n^{-1} \widetilde{T}_n \widetilde{Q}_n)^{-1} = (\widetilde{P}_n + \widetilde{T}_n^{-1} T_n Q_n). \tag{3.5}$$

Using Eqs. (3.4) and (3.5), we get

$$G_n = \widetilde{G}_n (\widetilde{P}_n + \widetilde{T}_n^{-1} T_n Q_n). \tag{3.6}$$

Relation (3.3) can be immediately obtained from Eqs. (3.4) and (3.6).
(ii) By virtue of (3.5) and (3.6), we see that $\operatorname{im} G_n = \operatorname{im} \widetilde{G}_n$. Additionally, observing that

$$\begin{aligned} B_n P_{n-1} z &= B_n \widetilde{P}_{n-1} P_{n-1} z + B_n \widetilde{Q}_{n-1} P_{n-1} z \\ &= B_n \widetilde{P}_{n-1} z + B_n T_n Q_n T_n^{-1} \widetilde{Q}_{n-1} P_{n-1} z = B_n \widetilde{P}_{n-1} z + G_n T_n^{-1} \widetilde{Q}_{n-1} P_{n-1} z, \end{aligned}$$

it follows that $B_n P_{n-1} z \in \operatorname{im} G_n$ iff $B_n \widetilde{P}_{n-1} z \in \operatorname{im} \widetilde{G}_n$. This means that the assertion (ii) is true.
(iii) For any $z_1 \in (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}) S_{1,n}$ there exists $z \in S_{1,n}$ such that $z_1 = (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}) z$, and therefore,

$$\begin{aligned} B_n P_{n-1} z_1 &= B_n P_{n-1} (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}) z = B_n P_{n-1} z + B_n P_{n-1} \widetilde{P}_{n-1} \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1} z \\ &= B_n P_{n-1} z + B_n P_{n-1} \widetilde{P}_{n-1} \widetilde{Q}_{n-1} \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1} z = B_n P_{n-1} z. \end{aligned}$$

Thus, we get

$$B_n P_{n-1} z_1 = B_n P_{n-1} z. \tag{3.7}$$

On the other hand, since $z \in S_{1,n}$, it follows $B_n P_{n-1} z \in \operatorname{im} G_n$, and taking into account (3.7) we obtain

$$B_n P_{n-1} z_1 \in \operatorname{im} G_n,$$

that is, $z_1 \in S_{1,n}$. Hence,

$$(\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}) S_{1,n} \subseteq S_{1,n}.$$

Next, take $z \in S_{1,n}$ be arbitrary and put $z_1 = (P_{n-1} + T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1})z$ we consider

$$\begin{aligned} B_n P_{n-1} z_1 &= B_n P_{n-1}(P_{n-1} + T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1})z \\ &= B_n P_{n-1} z + B_n P_{n-1} Q_{n-1} T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1} z = B_n P_{n-1} z. \end{aligned}$$

This leads to that $z_1 \in S_{1,n}$. Paying attention on (3.5), it is seen that $z = (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}) z_1$ with $z_1 \in S_{1,n}$, i.e., $z \in (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}) S_{1,n}$. Thus, we come to the conclusion

$$S_{1,n} \subseteq (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1}) S_{1,n}.$$

The proof of Lemma 3.3 is completed. $\square$

Applying Lemma 3.3, we obtain the identity

$$\widetilde{N}_{1,n-1} \cap \widetilde{S}_{1,n} = (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1} T_{n-1} Q_{n-1})(N_{1,n-1} \cap S_{1,n}). \tag{3.8}$$

It is worth noting that Eq. (3.8) guarantees that the following definition does not depend on the choice of the projections onto $N_n$ and the isomorphisms between $N_n$ and $N_{n-1}$. Let $G_{-1} := G_0$.

**Definition 3.4.** The LIDE (1.1) is said to be of index-2 tractable (index-2 for short) if

(i) $\dim(N_{n-1} \cap S_n) \equiv m - s,\ 1 \leqslant s \leqslant m-1$,
(ii) $N_{1,n-1} \cap S_{1,n} = \{0\}$ for all $n \geqslant 0$.

## 4. Solvability of index-2 LIDEs

From now on, we assume that the LIDE (1.1) is of index-2. Due to Corollary 3.2, we see that rank $G_n$ does not depend on the choice of the projections onto $N_n$ and the isomorphisms between $N_n$ and $N_{n-1}$. Suppose that rank $G_n \equiv s,\ 1 \leqslant s \leqslant m-1$. Let $Q_{1,n}$ be a projection onto $N_{1,n}$. Put $P_{1,n} := I - Q_{1,n}$. We also introduce for (1.1) an operator $T_{1,n} \in \mathrm{GL}(\mathbb{R}^m)$ satisfying $T_{1,n}|_{N_{1,n}}$ to be an isomorphism between $N_{1,n}$ and $N_{1,n-1}$, and a chain of matrices

$$G_{1,n} := G_n + B_n P_{n-1} T_{1,n} Q_{1,n}, \quad n \geqslant 0.$$

We recall that due to Lemma A.1, index-1 holds if and only if the matrices $G_n := A_n + B_n T_n Q_n$ are nonsingular. In the index-2 case, we get the following useful lemma.

**Lemma 4.1.** *The following assertions are equivalent*

(i) *the matrix* $G_{1,n} := G_n + B_n P_{n-1} T_{1,n} Q_{1,n}$ *is nonsingular*;
(ii) $N_{1,n-1} \oplus S_{1,n} = \mathbb{R}^m$;
(iii) $N_{1,n-1} \cap S_{1,n} = \{0\}$.

*Moreover, if* $G_{1,n}$ *is nonsingular then*

$$\widehat{Q}_{1,n-1} := T_{1,n} Q_{1,n} G_{1,n}^{-1} B_n P_{n-1} \tag{4.1}$$

*projects* $\mathbb{R}^m$ *onto* $N_{1,n-1}$ *along* $S_{1,n}$.

**Proof.** The proof is immediately deduced from Lemmas A.1 and A.2 by putting $A = G_n$, $B = B_n P_{n-1}$ and $\overline{A} = G_{n-1}$. $\square$

Let $\widehat{P}_{1,n} := I - \widehat{Q}_{1,n}$ with $\widehat{Q}_{1,n}$ given by (4.1).

**Remark 4.2**

1. By virtue of Lemmas 4.1 and 3.3, the matrices

   $G_{1,n} := G_n + B_nP_{n-1}T_{1,n}Q_{1,n} = A_n + B_nT_nQ_n + B_nP_{n-1}T_{1,n}Q_{1,n}$

   are nonsingular and the nonsingularity of $G_{1,n}$ does not depend on the choice of the projections $Q_n, Q_{1,n}$ and the isomorphisms $T_n, T_{1,n}$.
2. Applying Lemma 4.1 once more, we obtain

$$\widehat{Q}_{1,n-1}Q_{n-1} = 0, \tag{4.2}$$
$$\widehat{Q}_{1,n-1} = T_{1,n}\widehat{Q}_{1,n}\widehat{G}_{1,n}^{-1}B_nP_{n-1}, \tag{4.3}$$

   where $\widehat{G}_{1,n} := G_n + B_nP_{n-1}T_{1,n}\widehat{Q}_{1,n}$.

We now need the following lemma to solve IVPs for the index-2 LIDEs.

**Lemma 4.3.** *Suppose the LIDE* (1.1) *to be of index-*2. *There hold the following relations*

$$\text{(i)} \quad \widehat{G}_{1,n}^{-1}G_n = \widehat{P}_{1,n}, \quad \widehat{G}_{1,n}^{-1}A_n = \widehat{P}_{1,n}P_n; \tag{4.4}$$
$$\text{(ii)} \quad \widehat{G}_{1,n}^{-1}B_n = \widehat{G}_{1,n}^{-1}B_nP_{n-1}\widehat{P}_{1,n-1} + T_{1,n}^{-1}\widehat{Q}_{1,n-1} + \widehat{P}_{1,n}T_n^{-1}Q_{n-1}. \tag{4.5}$$

**Proof.** (i) From $G_n := A_n + B_nT_nQ_n$, it follows $G_nP_n = A_nP_n = A_n$. Thus, by using Lemma A.2 with $A = G_n$; $\overline{A} = G_{n-1}$ and $B = B_nP_{n-1}$ we have (4.4).

(ii) To prove (4.5), we rewrite $\widehat{G}_{1,n}^{-1}B_n$ as

$$\widehat{G}_{1,n}^{-1}B_n = \widehat{G}_{1,n}^{-1}B_nP_{n-1} + \widehat{G}_{1,n}^{-1}B_nQ_{n-1}. \tag{4.6}$$

Using again Lemma A.2 with $A = G_n, B = B_nP_{n-1}, \overline{A} = G_{n-1}$ and $\overline{Q} = \widehat{Q}_{1,n-1}$ we have

$$\widehat{G}_{1,n}^{-1}B_nP_{n-1} = \widehat{G}_{1,n}^{-1}B_nP_{n-1}\widehat{P}_{1,n-1} + T_{1,n}^{-1}\widehat{Q}_{1,n-1}. \tag{4.7}$$

Additionally, applying the formula (4.4) we come to

$$\begin{aligned}\widehat{G}_{1,n}^{-1}B_nQ_{n-1} &= \widehat{G}_{1,n}^{-1}B_nT_nT_n^{-1}Q_{n-1} = \widehat{G}_{1,n}^{-1}B_nT_nQ_nT_n^{-1}Q_{n-1}\\ &= \widehat{G}_{1,n}^{-1}(A_n + B_nT_nQ_n)Q_nT_n^{-1}Q_{n-1} = \widehat{G}_{1,n}^{-1}G_nQ_nT_n^{-1}Q_{n-1}\\ &= \widehat{P}_{1,n}T_n^{-1}Q_{n-1}.\end{aligned}$$

Hence,

$$\widehat{G}_{1,n}^{-1}B_nQ_{n-1} = \widehat{P}_{1,n}T_n^{-1}Q_{n-1}. \tag{4.8}$$

By (4.6) and taking into account (4.7) and (4.8), we obtain the relation (4.5). Lemma 4.3 is proved. □

We are now in the position of using the decomposition technique for index-2 LIDEs. Observing that

$$\mathbb{R}^m = \operatorname{im} Q_n \oplus \operatorname{im} P_n\widehat{P}_{1,n} \oplus \operatorname{im} P_n\widehat{Q}_{1,n},$$

we can decompose the solution $x_{n+1}$ of the index-2 LIDE (1.1) into

$$x_{n+1} = Q_n x_{n+1} + P_n \widehat{P}_{1,n} x_{n+1} + P_n \widehat{Q}_{1,n} x_{n+1} =: w_{n+1} + u_{n+1} + P_n v_{n+1}. \tag{4.9}$$

Scaling the index-2 LIDE (1.1) by $\widehat{G}_{1,n}^{-1}$ and applying Eqs. (4.4) and (4.5) we obtain

$$\widehat{P}_{1,n} P_n x_{n+1} = \widehat{G}_{1,n}^{-1} B_n P_{n-1} \widehat{P}_{1,n-1} x_n + T_{1,n}^{-1} \widehat{Q}_{1,n-1} x_n + \widehat{P}_{1,n} T_n^{-1} Q_{n-1} x_n + \widehat{G}_{1,n}^{-1} q_n. \tag{4.10}$$

From (4.2) we have $(I - \widehat{P}_{1,n}) Q_n = 0$, or $\widehat{P}_{1,n} Q_n = Q_n$ which implies $P_n \widehat{P}_{1,n} Q_n = 0$. On the other hand, observing that

$$\widehat{Q}_{1,n} T_{1,n}^{-1} \widehat{Q}_{1,n-1} = T_{1,n}^{-1} \widehat{Q}_{1,n-1}, \quad T_n^{-1} Q_{n-1} = Q_n T_n^{-1} Q_{n-1}, \tag{4.11}$$

it yields

$$P_n \widehat{P}_{1,n} \widehat{P}_{1,n} P_n = P_n \widehat{P}_{1,n} (P_n + Q_n) = P_n \widehat{P}_{1,n}, \quad P_n \widehat{P}_{1,n} T_{1,n}^{-1} \widehat{Q}_{1,n-1} = P_n \widehat{P}_{1,n} \widehat{Q}_{1,n} T_{1,n}^{-1} \widehat{Q}_{1,n-1} = 0,$$

and

$$P_n \widehat{P}_{1,n} \widehat{P}_{1,n} T_n^{-1} Q_{n-1} = P_n \widehat{P}_{1,n} Q_n T_n^{-1} Q_{n-1} = 0.$$

Thus, we come to the following relations

$$P_n \widehat{P}_{1,n} \widehat{P}_{1,n} P_n = P_n \widehat{P}_{1,n}, \quad P_n \widehat{P}_{1,n} T_{1,n}^{-1} \widehat{Q}_{1,n-1} = 0, \quad P_n \widehat{P}_{1,n} \widehat{P}_{1,n} T_n^{-1} Q_{n-1} = 0. \tag{4.12}$$

Using the relation (4.2) once again and Eq. (4.11), we see that

$$Q_n \widehat{P}_{1,n} \widehat{P}_{1,n} P_n = Q_n \widehat{P}_{1,n} P_n = Q_n (I - \widehat{Q}_{1,n}) P_n = -Q_n \widehat{Q}_{1,n} (I - Q_n) = -Q_n \widehat{Q}_{1,n},$$

and

$$\begin{aligned} &Q_n \widehat{P}_{1,n} T_{1,n}^{-1} \widehat{Q}_{1,n-1} = Q_n \widehat{P}_{1,n} \widehat{Q}_{1,n} T_{1,n}^{-1} \widehat{Q}_{1,n-1} = 0, \\ &Q_n \widehat{P}_{1,n} \widehat{P}_{1,n} T_n^{-1} Q_{n-1} = Q_n \widehat{P}_{1,n} Q_n T_n^{-1} Q_{n-1} = Q_n (I - \widehat{Q}_{1,n}) Q_n T_n^{-1} Q_{n-1} = T_n^{-1} Q_{n-1}. \end{aligned}$$

Hence,

$$Q_n \widehat{P}_{1,n} \widehat{P}_{1,n} P_n = -Q_n \widehat{Q}_{1,n}, \quad Q_n \widehat{P}_{1,n} T_{1,n}^{-1} \widehat{Q}_{1,n-1} = 0, \quad Q_n \widehat{P}_{1,n} \widehat{P}_{1,n} T_n^{-1} Q_{n-1} = T_n^{-1} Q_{n-1}. \tag{4.13}$$

Noting that (4.3) can be rewritten as

$$\widehat{Q}_{1,n} \widehat{G}_{1,n}^{-1} B_n P_{n-1} = T_{1,n}^{-1} \widehat{Q}_{1,n-1},$$

it follows

$$\widehat{Q}_{1,n} \widehat{G}_{1,n}^{-1} B_n P_{n-1} \widehat{P}_{1,n-1} = 0. \tag{4.14}$$

From the decomposition (4.9) and applying Eqs. (4.11)–(4.14), we see that Eq. (4.10) can be decoupled into three parts when multiplying it by $P_n \widehat{P}_{1,n}$, $Q_n \widehat{P}_{1,n}$ and $\widehat{Q}_{1,n}$, respectively

$$\begin{cases} u_{n+1} = P_n \widehat{P}_{1,n} \widehat{G}_{1,n}^{-1} B_n u_n + P_n \widehat{P}_{1,n} \widehat{G}_{1,n}^{-1} q_n, \\ -Q_n v_{n+1} = Q_n \widehat{P}_{1,n} \widehat{G}_{1,n}^{-1} B_n u_n + T_n^{-1} w_n + Q_n \widehat{P}_{1,n} \widehat{G}_{1,n}^{-1} q_n, \\ 0 = T_{1,n}^{-1} v_n + \widehat{Q}_{1,n} \widehat{G}_{1,n}^{-1} q_n. \end{cases}$$

Using Eq. (4.14) once more, we can rewrite the above system as follows

$$\begin{cases} u_{n+1} = P_n \widehat{G}_{1,n}^{-1} B_n u_n + P_n \widehat{P}_{1,n} \widehat{G}_{1,n}^{-1} q_n, \\ v_n = -T_{1,n} \widehat{Q}_{1,n} \widehat{G}_{1,n}^{-1} q_n, \\ w_n = -T_n Q_n v_{n+1} - T_n Q_n \widehat{G}_{1,n}^{-1} B_n u_n - T_n Q_n \widehat{P}_{1,n} \widehat{G}_{1,n}^{-1} q_n \end{cases} \tag{4.15}$$

for $n \geqslant 0$.

The first equation of (4.15) is an ordinary difference equation, called an inherent regular ordinary difference equation of the index-2 LIDE (1.1), and it can be solved by induction if $u_0$ is given. We set up an initial condition

$$P_0 \widehat{P}_{1,0} (x_0 - x^0) = 0, \tag{4.16}$$

which says that $u_0 = P_{-1}\widehat{P}_{1,-1}x_0 = P_0\widehat{P}_{1,0}x_0 = P_0\widehat{P}_{1,0}x^0$. Therefore, we can solve $u_n$ and substitute it into the last equation of (4.15) to get $w_n$. Hence, we obtain $x_n$ by using (4.9).

**Remark 4.4.** $\operatorname{im} P_{n-1}\widehat{P}_{1,n-1}$ is an invariant subspace of the ordinary difference equation (4.15) in the sense:

$$\text{if } u_0 \in \operatorname{im} P_{-1}\widehat{P}_{1,-1} = \operatorname{im} P_0\widehat{P}_{1,0}, \text{ then } u_n = P_{n-1}\widehat{P}_{1,n-1}u_n \text{ for all } n \geqslant 0.$$

Moreover, Eq. (4.15) also leads to $v_n = \widehat{Q}_{1,n-1}v_n$ and $w_n = Q_{n-1}w_n$ for all $n \geqslant 0$, respectively.

We are now interested in the case of a homogeneous equation (1.1)

$$A_n x_{n+1} = B_n x_n, \quad n \geqslant 0.$$

Using (4.15) we have

$$\begin{cases} u_{n+1} = P_n\widehat{G}_{1,n}^{-1}B_n u_n, \\ v_n = 0, \\ w_n = -T_n Q_n \widehat{G}_{1,n}^{-1}B_n u_n, \end{cases} \quad n \geqslant 0. \tag{4.17}$$

Recalling $x_n = P_{n-1}\widehat{P}_{1,n-1}x_n + Q_{n-1}x_n + P_{n-1}\widehat{Q}_{1,n-1}x_n = u_n + w_n + P_{n-1}v_n$ for all $n \geqslant 0$ implies that

$$x_{n+1} = \Pi_n \widehat{G}_{1,n}^{-1}B_n P_{n-1}\widehat{P}_{1,n-1}x_n, \quad n \geqslant 0, \tag{4.18}$$

here

$$\Pi_n := (I - T_{n+1}Q_{n+1}\widehat{G}_{1,n+1}^{-1}B_{n+1})P_n\widehat{P}_{1,n}. \tag{4.19}$$

Next, we observe that

$$\begin{aligned} P_{n-1}\widehat{P}_{1,n-1}\Pi_{n-1} &= P_{n-1}\widehat{P}_{1,n-1}(I - T_n Q_n \widehat{G}_{1,n}^{-1}B_n)P_{n-1}\widehat{P}_{1,n-1} \\ &= P_{n-1}\widehat{P}_{1,n-1}P_{n-1}\widehat{P}_{1,n-1} - P_{n-1}\widehat{P}_{1,n-1}Q_{n-1}T_n Q_n \widehat{G}_{1,n}^{-1}B_n P_{n-1}\widehat{P}_{1,n-1} \\ &= P_{n-1}\widehat{P}_{1,n-1} - P_{n-1}Q_{n-1}T_n Q_n \widehat{G}_{1,n}^{-1}B_n P_{n-1}\widehat{P}_{1,n-1} \\ &= P_{n-1}\widehat{P}_{1,n-1}. \end{aligned}$$

Hence,

$$P_{n-1}\widehat{P}_{1,n-1}\Pi_{n-1} = P_{n-1}\widehat{P}_{1,n-1}. \tag{4.20}$$

This relation implies that

$$\Pi_{n-1}^2 = (I - T_n Q_n \widehat{G}_{1,n}^{-1}B_n)P_{n-1}\widehat{P}_{1,n-1}\Pi_{n-1} = \Pi_{n-1},$$

i.e., $\Pi_{n-1}$ is a projection. In addition, it is easy to verify that

$$\Pi_n\widehat{G}_{1,n}^{-1}B_n P_{n-1}\widehat{P}_{1,n-1}\Pi_{n-1} = \Pi_n\widehat{G}_{1,n}^{-1}B_n P_{n-1}\widehat{P}_{1,n-1} = \Pi_n\widehat{G}_{1,n}^{-1}B_n P_{n-1} = \Pi_n\widehat{G}_{1,n}^{-1}B_n. \tag{4.21}$$

From the relations (4.19) and (4.20), we see that

$$\ker \Pi_{n-1} = \ker P_{n-1}\widehat{P}_{1,n-1}. \tag{4.22}$$

Now, let $x \in \ker P_{n-1}\widehat{P}_{1,n-1}$, i.e., $P_{n-1}\widehat{P}_{1,n-1}x = 0$. We write $x$ as

$$x = \widehat{P}_{1,n-1}x + \widehat{Q}_{1,n-1}x.$$

Obviously, $\widehat{Q}_{1,n-1}x$ lies in $N_{1,n-1}$ because $\widehat{Q}_{1,n-1}$ is a projection onto $N_{1,n-1}$. Further, since $P_{n-1}\widehat{P}_{1,n-1}x = 0$, it follows that

$$\widehat{P}_{1,n-1}x = (P_{n-1} + Q_{n-1})\widehat{P}_{1,n-1}x = Q_{n-1}\widehat{P}_{1,n-1}x \in N_{n-1}.$$

Therefore,

$$\ker P_{n-1}\widehat{P}_{1,n-1} \subseteq N_{n-1} + N_{1,n-1}.$$

Conversely, for arbitrary $x = y + z \in N_{n-1} + N_{1,n-1}$, we see that $P_{n-1}\widehat{P}_{1,n-1}x = P_{n-1}\widehat{P}_{1,n-1}Q_{n-1}y + P_{n-1}\widehat{P}_{1,n-1}\widehat{Q}_{1,n-1}z = 0$. Thus,

$$N_{n-1} + N_{1,n-1} \subseteq \ker P_{n-1}\widehat{P}_{1,n-1},$$

which yields that

$$\ker P_{n-1}\widehat{P}_{1,n-1} = N_{n-1} + N_{1,n-1}.$$

On the other hand, let $x \in N_{n-1} \cap N_{1,n-1}$ be arbitrary, it yields $Q_{n-1}x = x$ and $\widehat{Q}_{1,n-1}x = x$. Taking into account (4.2), we get

$$x = \widehat{Q}_{1,n-1}x = \widehat{Q}_{1,n-1}Q_{n-1}x = 0,$$

which says

$$N_{n-1} \cap N_{1,n-1} = \{0\}.$$

This leads to the following equality

$$\ker P_{n-1}\widehat{P}_{1,n-1} = N_{n-1} \oplus N_{1,n-1}.$$

Combining the above equality with (4.22), we obtain

$$\ker \Pi_{n-1} = N_{n-1} \oplus N_{1,n-1}. \tag{4.23}$$

Applying Corollary 3.2, we have that

$$\begin{aligned}\dim(\operatorname{im}\Pi_{n-1}) &= m - \dim(\ker \Pi_{n-1}) = m - \dim(N_{n-1} \oplus N_{1,n-1})\\ &= m - \dim N_{n-1} - \dim(N_{n-2} \cap S_{n-1}).\end{aligned}$$

Thus, we come to the conclusion that

$$\dim(\operatorname{im}\Pi_{n-1}) = m - \dim N_{n-1} - \dim(N_{n-2} \cap S_{n-1}).$$

We have two useful lemmas as follows.

**Lemma 4.5.** *There exist three vectors $x_n, x_{n+1}, x_{n+2} \in \mathbb{R}^m$ satisfying*

$$\begin{cases} A_n x_{n+1} = B_n x_n, \\ A_{n+1}x_{n+2} = B_{n+1}x_{n+1} \end{cases} \tag{4.24}$$

*if and only if $x_n \in \operatorname{im}\Pi_{n-1}$.*

**Proof.** Firstly, suppose there are three vectors $x_n$, $x_{n+1}$ and $x_{n+2}$ in $\mathbb{R}^m$ satisfying (4.24). By virtue of Eqs. (4.17) and (4.19) we obtain

$$x_n = u_n + w_n = (I - T_nQ_n\widehat{G}_{1,n}^{-1}B_n)u_n = (I - T_nQ_n\widehat{G}_{1,n}^{-1}B_n)P_{n-1}\widehat{P}_{1,n-1}u_n = \Pi_{n-1}u_n,$$

which implies $x_n \in \operatorname{im}\Pi_{n-1}$.

Now, let $x_n \in \operatorname{im}\Pi_{n-1}$, i.e., there exists a vector $\xi \in \mathbb{R}^m$ such that $x_n = \Pi_{n-1}\xi$. Put

$$x_{n+1} := \Pi_n\widehat{G}_{1,n}^{-1}B_n\xi; \quad x_{n+2} := \Pi_{n+1}\widehat{G}_{1,n+1}^{-1}B_{n+1}\Pi_n\widehat{G}_{1,n}^{-1}B_n\xi.$$

Applying the formulae (4.18) and (4.21) we see that

$$A_nx_{n+1} = A_n\Pi_n\widehat{G}_{1,n}^{-1}B_n\xi \overset{\text{by (4.21)}}{=} A_n\Pi_n\widehat{G}_{1,n}^{-1}B_nP_{n-1}\widehat{P}_{1,n-1}\Pi_{n-1}\xi \overset{\text{by (4.18)}}{=} B_n\Pi_{n-1}\xi = B_nx_n.$$

Similarly,

$$A_{n+1}x_{n+2} = B_{n+1}x_{n+1}.$$

The proof of lemma is completed. $\square$

**Lemma 4.6.** *The matrices $\Pi_{n-1}$ and $\Pi_n\widehat{G}_{1,n}^{-1}B_n$ are independent of the choice of $Q_n,\ T_n$ and $T_{1,n}$.*

**Proof.** Let $\widetilde{T}_n$ be another transformation satisfying $\widetilde{T}_n|_{\ker A_n}$ to be an isomorphism from $\ker A_n$ onto $\ker A_{n-1}$ and $\widetilde{Q}_n$ be another projection onto $\ker A_n$. Denote $\widetilde{G}_n := A_n + B_n\widetilde{T}_n\widetilde{Q}_n$. Let $\widehat{\widetilde{Q}}_{1,n}$ be a projection onto $\ker \widetilde{G}_n$ along $S_{1,n+1}$, $\widetilde{T}_{1,n}|_{\ker \widetilde{G}_n}$ denotes an isomorphism from $\ker \widetilde{G}_n$ onto $\ker \widetilde{G}_{n-1}$ and put

$$\widehat{\widetilde{G}}_{1,n} := \widetilde{G}_n + B_n\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}, \quad \widetilde{\Pi}_{n-1} := (I - \widetilde{T}_n\widetilde{Q}_n\widehat{\widetilde{G}}_n^{-1}B_n)\widetilde{P}_{n-1}\widehat{\widetilde{P}}_{1,n-1}.$$

By Lemma 4.5, it follows that $\operatorname{im}\Pi_{n-1}$ does not depend on the choice of $Q_n,\ T_n$ and $T_{1,n}$. Hence, we only need to show that $\ker \Pi_{n-1}$ is independent of the choice of $Q_n,\ T_n$ and $T_{1,n}$. Thanks to Eq. (4.23), it is equivalent to prove that

$$\ker A_{n-1} \oplus \ker G_{n-1} = \ker A_{n-1} \oplus \ker \widetilde{G}_{n-1}.$$

Let $x \in \ker G_{n-1}$ be arbitrary. Using the formula (3.3) in Lemma 3.3 we have

$$(\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1})x \in \ker \widetilde{G}_{n-1}.$$

Observe that

$$\begin{aligned} x &= (I - \widetilde{P}_{n-1} - \widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1})x + (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1})x \\ &= (\widetilde{Q}_{n-1}x - \widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1}x) + (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1})x. \end{aligned}$$

Further, it is clear that $A_{n-1}\widetilde{Q}_{n-1}x = 0$ and $A_{n-1}\widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1}x = 0$, i.e., $\widetilde{Q}_{n-1}x - \widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1}x \in \ker A_{n-1}$. Thus, $\ker G_{n-1} \subseteq \ker A_{n-1} \oplus \ker \widetilde{G}_{n-1}$, which follows that

$$\ker A_{n-1} \oplus \ker G_{n-1} \subseteq \ker A_{n-1} \oplus \ker \widetilde{G}_{n-1}.$$

Similarly, we obtain the inverse inclusion. Therefore,

$$\ker A_{n-1} \oplus \ker G_{n-1} = \ker A_{n-1} \oplus \ker \widetilde{G}_{n-1}.$$

We show that $\Pi_n\widehat{G}_{1,n}^{-1}\widehat{\widetilde{G}}_{1,n} = \widetilde{\Pi}_n$. Indeed, we have

$$\begin{aligned} \Pi_n\widehat{G}_{1,n}^{-1}\widehat{\widetilde{G}}_{1,n} &= \Pi_n\widehat{G}_{1,n}^{-1}(A_n + B_n\widetilde{T}_n\widetilde{Q}_n + B_n\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}) \\ &= \Pi_n\widehat{G}_{1,n}^{-1}A_n + \Pi_n\widehat{G}_{1,n}^{-1}B_nQ_{n-1}\widetilde{T}_n\widetilde{Q}_n \\ &\quad +\Pi_n\widehat{G}_{1,n}^{-1}B_nP_{n-1}\widehat{P}_{1,n-1}\widetilde{P}_{n-1}\widehat{\widetilde{Q}}_{1,n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}. \end{aligned}$$

From (4.4) it implies $\Pi_n\widehat{G}_{1,n}^{-1}A_n = \Pi_n\widehat{P}_{1,n}P_n = \Pi_n$. On the other hand, multiplying both sides of (4.8) by $\Pi_n$, it yields

$$\begin{aligned} \Pi_n\widehat{G}_{1,n}^{-1}B_nQ_{n-1} &= \Pi_n\widehat{P}_{1,n}T_n^{-1}Q_{n-1} = \Pi_nP_n\widehat{P}_{1,n}\widehat{P}_{1,n}T_n^{-1}Q_{n-1} \\ &= \Pi_nP_n(I - \widehat{Q}_{1,n})Q_nT_n^{-1}Q_{n-1} = 0. \end{aligned}$$

Since $\operatorname{im}\widetilde{Q}_{n-1} = \ker A_{n-1} \subseteq \ker P_{n-1}\widehat{P}_{1,n-1}$ and $\operatorname{im}\widehat{\widetilde{Q}}_{1,n-1} = \ker \widetilde{G}_{n-1} \subseteq \ker P_{n-1}\widehat{P}_{1,n-1}$, it follows that

$$P_{n-1}\widehat{P}_{1,n-1}\widetilde{P}_{n-1}\widehat{\widetilde{Q}}_{1,n-1} = P_{n-1}\widehat{P}_{1,n-1}(I - \widetilde{Q}_{n-1})\widehat{\widetilde{Q}}_{1,n-1} = 0.$$

Therefore, we come to the relation $\Pi_n\widehat{G}_{1,n}^{-1}\widehat{\widetilde{G}}_{1,n} = \Pi_n$, or equivalently, $\Pi_n\widehat{G}_{1,n}^{-1}\widehat{\widetilde{G}}_{1,n} = \widetilde{\Pi}_n$, i.e., $\Pi_n\widehat{G}_{1,n}^{-1} = \widetilde{\Pi}_n\widehat{\widetilde{G}}_{1,n}^{-1}$. Thus, $\Pi_n\widehat{G}_{1,n}^{-1}B_n$ does not depend on the choice of $Q_n,\ T_n$ and $T_{1,n}$. Lemma 4.6 is proved. $\square$

Summing up, we obtain the main result of this paper.

**Theorem 4.7.** *Let the LIDE (1.1) be of index-2. Then, the following assertions hold.*

(i) *The subspace* $\operatorname{im} \Pi_{n-1}$ *describes the solution space of the homogeneous equation. Moreover, the solution space* $\operatorname{im} \Pi_{n-1}$ *is a proper subspace of* $S_n$ *and*

$$\dim(\operatorname{im} \Pi_{n-1}) = m - \dim N_{n-1} - \dim(N_{n-2} \cap S_{n-1}).$$

(ii) *The IVP (1.1) and (4.16) has a unique solution*

$$\begin{aligned} x_n = \Pi_{n-1}u_n + T_nQ_nT_{1,n+1}\widehat{Q}_{1,n+1}\widehat{G}_{1,n+1}^{-1}q_{n+1} \\ -T_nQ_n\widehat{P}_{1,n}\widehat{G}_{1,n}^{-1}q_n - P_{n-1}T_{1,n}\widehat{Q}_{1,n}\widehat{G}_{1,n}^{-1}q_n, \quad n \geqslant 0, \end{aligned} \tag{4.25}$$

*where* $\{u_n\}$ *is a solution of the IVP for the inherent regular ordinary difference equation of the index-2 LIDE (1.1)*

$$u_{n+1} = P_n\widehat{G}_{1,n}^{-1}B_nu_n + P_n\widehat{P}_{1,n}\widehat{G}_{1,n}^{-1}q_n$$

*and*

$$u_0 = u^0 := P_0\widehat{P}_{1,0}x^0.$$

## 5. MET theorem for index-2 LIDEs

In this section, we apply ideas mentioned in Section 3 and Section 4 to study the cocycle property of the solutions and the Lyapunov spectrum for real noise index-2 LIDEs. Let $(\Omega, \mathcal{F}, P)$ be a probability space satisfying the normal conditions (cf. [10]) and let $\theta : (\Omega, \mathcal{F}, P) \to (\Omega, \mathcal{F}, P)$ be a $P$-preserving transformation. Giving two random variables $A(\cdot)$ and $B(\cdot)$ valued in the space of $m \times m$-matrices, we consider the equation

$$\begin{cases} A(\theta^n\omega)x_{n+1}(\omega) = B(\theta^n\omega)x_n(\omega), \quad n = 0, \pm 1, \pm 2, \ldots, \\ x_0 = x \in \mathbb{R}^m \text{ a.s.,} \end{cases} \tag{5.1}$$

where $\theta^n = \theta \circ \theta^{n-1}$.

### *5.1. Solutions of (5.1) for $n \geqslant 0$*

Henceforth, we assume that $\operatorname{rank} A(\omega) = r$ for $P$-a.s. $\omega \in \Omega$ where $r$ $(1 \leqslant r \leqslant m-1)$ is a nonrandom constant. Denote $N(\omega) := \ker A(\omega)$. Let $Q(\omega)$ be a measurable projection onto $N(\omega)$ and $T(\omega)$ be a random variable with values in $\mathrm{GL}(\mathbb{R}^m)$ such that $T(\omega)|_{N(\omega)}$ is an isomorphism from $N(\omega)$ onto $N(\theta^{-1}\omega)$ for all $\omega \in \Omega$. We can give such a projector $Q(\omega)$ and such a $T(\omega)$ by the following way: let matrix $A(\omega)$ with a constant rank $A(\omega) \equiv r$ possess a singular value decomposition

$$A(\omega) = U(\omega)\Sigma(\omega)V^\top(\omega),$$

where $U(\omega)$, $V(\omega)$ are orthogonal matrices and $\Sigma(\omega)$ is a diagonal matrix with singular values $\sigma_1(\omega) \geqslant \sigma_2(\omega) \geqslant \cdots \geqslant \sigma_r(\omega) > 0$ on its main diagonal. Since $A(\omega)$ is measurable, on the above decomposition of $A(\omega)$ we can choose the matrix $V(\omega)$ to be measurable (see the proof of Lemma 3 in [9]). Hence, we can put $Q(\omega) = V(\omega)\operatorname{diag}(O, I_{m-r})V^\top(\omega)$ and $T(\omega) = V(\theta^{-1}\omega)V^\top(\omega)$.

Let $P(\omega) := I - Q(\omega)$ and

$$S(\omega) := \{z : B(\omega)z \in \operatorname{im} A(\omega)\}, \quad G(\omega) := A(\omega) + B(\omega)T(\omega)Q(\omega).$$

We are going to define an index-2 tractable concept for Eq. (5.1) with $n \geqslant 0$. For this purpose, suppose that $G(\omega)$ is singular with probability 1. Denote

$$N_{1,0}(\omega) := \ker G(\omega), \quad S_{1,0}(\omega) := \{z \in \mathbb{R}^m : B(\omega)P(\theta^{-1}\omega)z \in \operatorname{im} G(\omega)\}.$$

Definition 3.4 for Eq. (5.1) with $n \geqslant 0$ is now read as follows: the LIDE (5.1) is said to be index-2 tractable if

$$\dim(N(\theta^{-1}\omega) \cap S(\omega)) = m - s(1 \leqslant s \leqslant m-1), \quad N_{1,0}(\theta^{-1}\omega) \cap S_{1,0}(\omega) = \{0\}$$

for a.s. $\omega \in \Omega$.

We remark that $\operatorname{rank} G(\omega) \equiv s$ for a.s. $\omega \in \Omega$. Let $\widehat{Q}_{1,0}(\omega)$ be a projection onto $N_{1,0}(\omega)$ along $S_{1,0}(\theta\omega)$. By (4.1) it is seen that $\widehat{Q}_{1,0}(\omega)$ is measurable. Put $\widehat{P}_{1,0} := I - \widehat{Q}_{1,0}$. We also introduce for Eq (5.1) a random variable $T_{1,0}$ with value in $\mathrm{GL}(\mathbb{R}^m)$ such that $T_{1,0}(\omega)|_{N_{1,0}(\omega)}$ is an isomorphism between $N_{1,0}(\omega)$ and $N_{1,0}(\theta^{-1}\omega)$, and denote

$$\widehat{G}_{1,0}(\omega) := G(\omega) + B(\omega)P(\theta^{-1}\omega)T_{1,0}(\omega)\widehat{Q}_{1,0}(\omega).$$

For the sake of simplicity we put

$$A_n(\omega) = A(\theta^n\omega),\ B_n(\omega) = B(\theta^n\omega) \text{ and so on.}$$

With these notations, Eq. (5.1) for $n \geqslant 0$ can be rewritten as the form

$$\begin{cases} A_n(\omega)x_{n+1}(\omega) = B_n(\omega)x_n(\omega), & n = 0, 1, \ldots, \\ x_0 = x \in \mathbb{R}^m \text{ a.s.} \end{cases} \tag{5.2}$$

Applying the solution expression formula (4.18) and paying attention on Eq. (4.21), we obtain that the IVP (5.2) has a unique solution

$$x_n(\omega) = \Pi_{n-1}\left(\prod_{i=n-1}^{0} \widehat{G}_{1,i}^{-1}B_i\right)(\omega)x, \quad n = 1, 2, \ldots,\ x_0(\omega) = x \tag{5.3}$$

provided $x \in \operatorname{im} \Pi_{-1}$ with probability 1.

*5.2. Solution of (5.1) for $n < 0$.*

In this case, Eq. (5.1) turns into the difference equation

$$\begin{cases} B_n(\omega)x_n(\omega) = A_n(\omega)x_{n+1}(\omega), & n = -1, -2, \ldots, \\ x_0 = x \in \mathbb{R}^m \text{ a.s.} \end{cases} \tag{5.4}$$

For (5.4), we suppose that $\operatorname{rank} B(\omega) = k$ for $P$-a.s. $\omega \in \Omega$ where $k$ $(1 \leqslant k \leqslant m-1)$ is a nonrandom constant. Denote

$$\begin{aligned} &\overline{N}(\omega) := \ker B(\omega), \quad \overline{S}(\omega) := \{z \in \mathbb{R}^m : A(\omega)z \in \operatorname{im} B(\omega)\}, \\ &\overline{G}(\omega) := B(\omega) + A(\omega)\overline{T}(\omega)\overline{Q}(\omega), \quad \overline{N}_{1,0}(\omega) := \ker \overline{G}(\omega), \\ &\overline{S}_{1,0}(\omega) := \{z \in \mathbb{R}^m : A(\omega)\overline{P}(\theta\omega)z \in \operatorname{im} \overline{G}(\omega)\}, \end{aligned}$$

where $\overline{Q}(\omega)$ is a measurable projection onto $\overline{N}(\omega)$, $\overline{P} := I - \overline{Q}$, and $\overline{T}$ is a random variable with value in $\mathrm{GL}(\mathbb{R}^m)$ such that $\overline{T}(\omega)|_{\overline{N}(\omega)}$ is an isomorphism from $\overline{N}(\omega)$ onto $\overline{N}(\theta\omega)$. We also assume that $\overline{G}(\omega)$ is singular with probability 1. Let Eq. (5.4) be index-2 tractable, i.e.,

$$\dim\left(\overline{N}(\theta\omega) \cap \overline{S}(\omega)\right) = m - t\ (1 \leqslant t \leqslant m-1), \quad \overline{N}_{1,0}(\theta\omega) \cap \overline{S}_{1,0}(\omega) = \{0\}$$

for a.s. $\omega \in \Omega$. Let $\widehat{\overline{Q}}_{1,0}(\omega)$ be a projection onto $\overline{N}_{1,0}(\omega)$ along $\overline{S}_{1,0}(\theta^{-1}\omega)$, $\widehat{\overline{P}}_{1,0} := I - \widehat{\overline{Q}}_{1,0}$ and $\overline{T}_{1,0} \in \mathrm{GL}(\mathbb{R}^m)$ be a random variable such that $\overline{T}_{1,0}(\omega)|_{\overline{N}_{1,0}(\omega)}$ is an isomorphism between $\overline{N}_{1,0}(\omega)$ and $\overline{N}_{1,0}(\theta\omega)$. We set

$$\widehat{\overline{G}}_{1,0}(\omega) := \overline{G}(\omega) + A(\omega)\overline{P}(\theta\omega)\overline{T}_{1,0}(\omega)\widehat{\overline{Q}}_{1,0}(\omega).$$

By the same argument as in the case $n \geqslant 0$ (cf. (5.3)), we can easily obtain the unique solution of (5.4) given by

$$x_n(\omega) = \overline{\Pi}_n \left( \prod_{i=n}^{-1} \widehat{\overline{G}}_{1,i}^{-1} A_i \right) (\omega)x, \quad n = -1, -2, \ldots, \quad x_0(\omega) = x, \tag{5.5}$$

provided that $x \in \operatorname{im} \overline{\Pi}_0$ with probability 1, where $\overline{\Pi}_n = (I - \overline{T}_{n-1}\overline{Q}_{n-1}\widehat{\overline{G}}_{1,n-1}^{-1}A_{n-1})\overline{P}_n\widehat{\overline{P}}_{1,n}$.

**Remark 5.1.** In Sections 5.3 and 5.4,

1. if $G(\omega)$ (resp. $\overline{G}(\omega)$) is nonsingular but $A(\omega)$ (resp. $B(\omega)$) is singular with a probability 1 then we obtain the index-1 concept for $n \geqslant 0$ dealt with in [4] (resp. for $n < 0$). In this case we choose

   $\widehat{Q}_{1,0}(\omega) = 0$ (resp. $\widehat{\overline{Q}}_{1,0}(\omega) = 0$);
2. if $A(\omega)$ (resp. $B(\omega)$) is nonsingular with probability 1, we say the system (5.1) to be index-0 for $n \geqslant 0$ (resp. $n < 0$). In this case we take

   $Q(\omega) = 0, \quad \widehat{Q}_{1,0}(\omega) = 0$ (resp. $\overline{Q}(\omega) = 0, \ \widehat{\overline{Q}}_{1,0}(\omega) = 0$).

Thus, in a generalized sense, we can consider the index-0 and index-1 are special cases of the index-2.

In what follows, assume that Eq. (5.1) is index-2 tractable in the generalized sense.

*5.3. Cocycle property of the solutions*

The following lemma plays an important role in investigating cocycle property of the solutions of (5.1).

**Lemma 5.2.** *The projections $\Pi_{-1}$ and $\overline{\Pi}_0$ commute, i.e.,*

$$\Pi_{-1}\overline{\Pi}_0 = \overline{\Pi}_0\Pi_{-1}$$

*with probability 1.*

**Proof.** Let $x \in \ker B_0$. Putting $x_0 := x, \ x_1 := 0, \ x_2 := 0$ we have

$$\begin{cases} A_0x_1 = B_0x_0, \\ A_1x_2 = B_1x_1. \end{cases}$$

By virtue of Lemma 4.5, it follows that $x \in \operatorname{im} \Pi_{-1}$. This leads to

$$\ker B_0 \subseteq \operatorname{im} \Pi_{-1}. \tag{5.6}$$

On the other hand, suppose that $x \in \ker \overline{G}_0$, then $(B_0 + A_0\overline{T}_0\overline{Q}_0)x = 0$, or equivalently, $A_0(-\overline{T}_0\overline{Q}_0x) = B_0x$. Further, $B_1\overline{T}_0\overline{Q}_0 = B_1\overline{Q}_1\overline{T}_0\overline{Q}_0 = 0$ which implies $B_1(-\overline{T}_0\overline{Q}_0x) = 0$. Putting $x_0 := x, \ x_1 := -\overline{T}_0\overline{Q}_0x, \ x_2 := 0$ we get

$$\begin{cases} A_0x_1 = B_0x_0, \\ A_1x_2 = B_1x_1. \end{cases}$$

Using Lemma 4.5 once more, we come to the conclusion that $x \in \operatorname{im} \Pi_{-1}$. Therefore, we obtain the following inclusion

$$\ker \overline{G}_0 \subseteq \operatorname{im} \Pi_{-1}. \tag{5.7}$$

Combining the inclusions (5.6) and (5.7), and using the fact $\ker \overline{\Pi}_0 = \ker B_0 \oplus \ker \overline{G}_0$, we have

$$\ker \overline{\Pi}_0 \subseteq \operatorname{im} \Pi_{-1}.$$

From the above inclusion it is now straightforward to get the following equality

$$(I - \Pi_{-1})(I - \overline{\Pi}_0) = 0.$$

Analogously,

$$(I - \overline{\Pi}_0)(I - \Pi_{-1}) = 0.$$

Thus, the projections $\Pi_{-1}$ and $\overline{\Pi}_0$ commute with probability 1. □

Next, we define

$$\Phi(n, \omega) = \begin{cases} \Pi_{n-1}\left(\prod_{i=n-1}^{0} \widehat{G}_{1,i}^{-1} B_i\right) \overline{\Pi}_0(\omega), & \text{if } n > 0, \\ \Pi_{-1}\overline{\Pi}_0(\omega), & \text{if } n = 0, \\ \overline{\Pi}_n \left(\prod_{i=n}^{-1} \widehat{\widehat{G}}_{1,i}^{-1} A_i\right) \Pi_{-1}(\omega), & \text{if } n < 0. \end{cases} \tag{5.8}$$

Denote $x_n(\omega; \bar{x}(\omega))$ the solution of (5.1) satisfying $x_0(\omega; \bar{x}(\omega)) = \bar{x}(\omega)$. By virtue of (4.21) it follows that

$$\Phi(n, \omega)x = \Pi_{n-1}\left(\prod_{i=n-1}^{0} \widehat{G}_{1,i}^{-1} B_i\right) \overline{\Pi}_0(\omega)x = \Pi_{n-1}\left(\prod_{i=n-1}^{0} \widehat{G}_{1,i}^{-1} B_i\right) \Pi_{-1}\overline{\Pi}_0(\omega)x \quad \text{if } n > 0,$$

$$\Phi(n, \omega)x = \overline{\Pi}_n\left(\prod_{i=n}^{-1} \widehat{\widehat{G}}_{1,i}^{-1} A_i\right) \Pi_{-1}(\omega) = \overline{\Pi}_n\left(\prod_{i=n}^{-1} \widehat{\widehat{G}}_{1,i}^{-1} A_i\right) \overline{\Pi}_0\Pi_{-1}(\omega)x \quad \text{if } n < 0.$$

Therefore, by mean of the formulae (5.3) and (5.5) we get

$$x_n(\omega; \bar{x}(\omega)) = \Phi(n, \omega)x \quad \text{with } \bar{x}(\omega) = (\Pi_{-1}\overline{\Pi}_0)(\omega)x.$$

Hence, for all $x \in \mathbb{R}^m$ we have

$$\begin{cases} A_n(\cdot)\Phi(n+1, \cdot)x = B_n\Phi(n, \cdot)x, \\ A_{n+1}(\cdot)\Phi(n+2, \cdot)x = B_{n+1}\Phi(n+1, \cdot)x. \end{cases}$$

Lemma 4.5 ensures that

$$\Phi(n, \cdot)x \in \operatorname{im} \Pi_{n-1} \text{ for all } x \in \mathbb{R}^m, \quad n = 0, \pm 1, \pm 2, \ldots$$

Consequently,

$$\Pi_{n-1}(\cdot)\Phi(n, \cdot) = \Phi(n, \cdot), \quad n = 0, \pm 1, \pm 2, \ldots$$

Obviously, the same equalities for $\overline{\Pi}_n$ are formed, i.e.,

$$\overline{\Pi}_n(\cdot)\Phi(n, \cdot) = \Phi(n, \cdot), \quad n = 0, \pm 1, \pm 2, \ldots$$

Therefore, the relation (5.8) can be rewritten as

$$\Phi(n, \omega) = \begin{cases} \left(\prod_{i=n-1}^{0} \Pi_i\widehat{G}_{1,i}^{-1} B_i\overline{\Pi}_i\right)(\omega), & \text{if } n > 0, \\ \Pi_{-1}\overline{\Pi}_0(\omega), & \text{if } n = 0, \\ \left(\prod_{i=n}^{-1} \overline{\Pi}_i\widehat{\widehat{G}}_{1,i}^{-1} A_i\Pi_i\right)(\omega), & \text{if } n < 0. \end{cases} \tag{5.9}$$

We are now in the position of giving a fundamental expression in random dynamical theory, called cocycle property (see [2]):

**Theorem 5.3.** *For any $q, p \in \mathbb{Z}$ the following relation holds*

$$\Phi(p + q, \omega) = \Phi(p, \theta^q\omega)\Phi(q, \omega). \tag{5.10}$$

**Proof.** If $q, p \leqslant 0$ or $q, p \geqslant 0$, using the fact (5.9) and properties of random matrix products, we can easily obtain Eq. (5.10). Let $q < 0 < p$. Firstly, we show that

$$\Phi(0, \omega) = \Phi(-1, \theta\omega)\Phi(1, \omega) \text{ for } P\text{-a.s. } \omega \in \Omega.$$

Indeed, it is worth mentioning that $A_0\Phi(1,\omega) = B_0\Phi(0,\omega)$, $\widehat{\widehat{G}}_{1,0}^{-1}B_0 = \widehat{\widehat{P}}_{1,0}\overline{P}_0$ (cf. (4.4)), and by definition we get

$$\Phi(-1,\theta\omega)\Phi(1,\omega) = \overline{\Pi}_0\widehat{\widehat{G}}_{1,0}^{-1}A_0\Pi_0\Phi(1,\omega) = \overline{\Pi}_0\widehat{\widehat{G}}_{1,0}^{-1}A_0\Phi(1,\omega) = \overline{\Pi}_0\widehat{\widehat{G}}_{1,0}^{-1}B_0\Phi(0,\omega)$$
$$= \overline{\Pi}_0\widehat{\widehat{P}}_{1,0}\overline{P}_0\Phi(0,\omega) = \overline{\Pi}_0\Phi(0,\omega) = \Phi(0,\omega).$$

Using this relation we see that

$$\Phi(p,\theta^q\omega)\Phi(q,\omega) = (\Phi(p-1,\theta^{q+1}\omega)\Phi(1,\theta^q\omega))(\Phi(-1,\theta^{q+1}\omega)\Phi(q+1,\omega))$$
$$= \Phi(p-1,\theta^{q+1}\omega)\Phi(0,\theta^{q+1}\omega)\Phi(q+1,\omega) = \Phi(p-1,\theta^{q+1}\omega)\Phi(q+1,\omega).$$

Continuing this way, we obtain $\Phi(p,\theta^q\omega)\Phi(q,\omega) = \Phi(p+q,\omega)$. The case $q > 0 > p$ is proved similarly. Theorem 5.3 is proved. $\square$

*5.4. Multiplicative ergodic theorem*

This section is concerned with Lyapunov exponents of solution of (5.1). Suppose that $\theta$ is an ergodic transformation on $(\Omega,\mathcal{F},P)$ and the following condition is satisfied

**Hypotheses 5.4**

$$\ln\|\Pi_0\widehat{G}_{1,0}^{-1}B_0\overline{\Pi}_0\| \in L_1(\Omega,\mathcal{F},P) \quad \text{and} \quad \ln\|\overline{\Pi}_0\widehat{\widehat{G}}_{1,0}^{-1}A_0\Pi_0\| \in L_1(\Omega,\mathcal{F},P). \tag{5.11}$$

Noting that these assumptions are independent of the choice of $T,\overline{T}$; $Q,\overline{Q}$; and $T_{1,0},\overline{T}_{1,0}$. Based on the fact $\Phi(n)$ is the product of ergodic stationary matrices $\Pi_n\widehat{G}_{1,n}^{-1}B_n\overline{\Pi}_n$ for $n > 0$ and $\overline{\Pi}_n\widehat{\widehat{G}}_{1,n}^{-1}A_n\Pi_n$ for $n < 0$, similar to the index-1 case (see [4] and therein references) we get

(i) Under the assumption (5.11), there exist the limits

$$\lim_{n\to+\infty}(\Phi(n,\omega)^\top\Phi(n,\omega))^{1/2n} =: \Delta(\omega) \quad \text{and} \quad \lim_{n\to-\infty}\left(\Phi(n,\omega)^\top\Phi(n,\omega)\right)^{1/2|n|} =: \overline{\Delta}(\omega).$$

(ii) Let $0 < e^{\lambda_1} < e^{\lambda_2} < \cdots < e^{\lambda_\tau}$ be the different nonzero eigenvalues of $\Delta$ and $\lambda_0 = -\infty$. We denote $\mathcal{U}_0 := \ker\Delta(\omega)$, and $\mathcal{U}_i,\ i = 1,\dots,\tau$ the eigenspaces with multipliers $d_i := \dim\mathcal{U}_i$ corresponding to the eigenvalues $e^{\lambda_i}$. Then, $\tau;\ d_i, \lambda_i,\ i = 1,\dots,\tau$ are nonrandom constants. Let $\mathcal{V}_k := \mathcal{U}_0 \oplus \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k,\ k = 0,\dots,\tau$ such that

$$\{0\} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_\tau = \mathbb{R}^m$$

defines a filtration of $\mathbb{R}^m$. For each $x \in \mathbb{R}^m$ the Lyapunov exponent

$$\lambda(\omega,x) = \lim_{n\to+\infty}\frac{1}{n}\ln\|\Phi(n,\omega)x\|$$

exists and

$$\lambda(\omega,x) = \lambda_k(\omega) \text{ iff } x \in (\mathcal{V}_k \setminus \mathcal{V}_{k-1})(\omega), \quad k = 1,\dots,\tau.$$

Further, $\lambda(\omega,x) = -\infty$ iff $x \in \mathcal{V}_0$. In addition, the spaces $\mathcal{V}_i$ are invariant in the sense

$$\Phi(n,\omega)\mathcal{V}_i(\omega) \subset \mathcal{V}_i(\theta^n\omega) \quad \text{for any } n \geqslant 0 \text{ and } i = 0,\dots,\tau.$$

(iii) A similar result can be formulated for the case $n \to -\infty$. That is, let $0 < e^{\overline{\lambda}_{\overline{\tau}}} < e^{\overline{\lambda}_{\overline{\tau}-1}} < \cdots < e^{\overline{\lambda}_1}$ be the different nonzero eigenvalues of $\overline{\Delta}$, and $\overline{\mathcal{U}}_{\overline{\tau}},\dots,\overline{\mathcal{U}}_1$ the corresponding eigenspaces with multipliers $\overline{d}_i := \dim\overline{\mathcal{U}}_i,\ i = \overline{\tau},\dots,1$. Denote $\overline{\lambda}_{\overline{\tau}+1} = -\infty$ and $\overline{\mathcal{U}}_{\overline{\tau}+1} := \ker\overline{\Delta}(\omega)$. Then $\overline{\tau};\ \overline{d}_i,\ \overline{\lambda}_i,\ i = 1,\dots,\overline{\tau}$ are nonrandom constants. Let $\overline{\mathcal{V}}_k = \overline{\mathcal{U}}_{\overline{\tau}+1} \oplus \overline{\mathcal{U}}_{\overline{\tau}} \oplus \cdots \oplus \overline{\mathcal{U}}_k,\ k = \overline{\tau}+1,\dots,1$, such that

$$\{0\} \subset \overline{\mathcal{V}}_{\overline{\tau}+1} \subset \overline{\mathcal{V}}_{\overline{\tau}} \subset \cdots \subset \overline{\mathcal{V}}_1 = \mathbb{R}^m$$

defines another filtration of $\mathbb{R}^m$. For any $x \in \mathbb{R}^m$ the Lyapunov exponent

$$\overline{\lambda}(\omega, x) = \lim_{n \to -\infty} \frac{1}{|n|} \ln \|\Phi(n, \omega)x\|$$

exists. In this case, we have that

$$\overline{\lambda}(\omega, x) = \overline{\lambda}_k(\omega) \text{ iff } x \in (\overline{\mathcal{V}}_k \setminus \overline{\mathcal{V}}_{k+1})(\omega), \quad k = 1, \ldots, \overline{\tau},$$

and

$$\overline{\lambda}(\omega, x) = -\infty \text{ iff } x \in \overline{\mathcal{V}}_{\overline{\tau}+1}.$$

Moreover, the spaces $\overline{\mathcal{V}}_i$ are invariant in the sense

$$\Phi(n, \omega)\overline{\mathcal{V}}_i(\omega) \subset \overline{\mathcal{V}}_i(\theta^n \omega), \quad n < 0, \ i = 1, \ldots, \overline{\tau} + 1.$$

Now we come to the multiplicative ergodic theorem MET for index-2 LIDEs. The proof of this theorem is omitted here due to lack of space, but is similar to the one in [4].

**Theorem 5.5.** *Let the LIDE (5.1) be of generalized index-2 and suppose that*

$$\ln \|\Pi_0 \widehat{G}_{1,0}^{-1} B_0 \overline{\Pi}_0\|, \ \ln \|\overline{\Pi}_0 \widehat{\overline{G}}_{1,0}^{-1} A_0 \Pi_0\| \in L_1(\Omega, \mathcal{F}, P).$$

*Then,*

(i) $\tau = \overline{\tau}, \quad \lambda_i = -\overline{\lambda}_i, \ d_i = \bar{d}_i$ *for* $i = 1, \ldots, \tau$ *and they are nonrandom numbers*;
(ii) *for any* $i = 1, \ldots, \tau$, *the set* $\mathcal{V}_i \cap \overline{\mathcal{V}}_i$ *is invariant*;
(iii) *there exist subspaces* $W_0, \ldots, W_\tau$ *such that* $W_0 = \mathcal{V}_0$ *and*

$$\mathbb{R}^m = \bigoplus_{i=0}^{\tau} W_i, \quad W_i \oplus \mathcal{V}_0 = \mathcal{V}_i \cap \overline{\mathcal{V}}_i, \quad \dim W_i = d_i, \quad i = 1, \ldots, \tau$$

*and for any* $x \in W_i \setminus \{0\}$

$$\lim_{n \to \pm\infty} \frac{1}{n} \ln \|\Phi(n, \omega)x(\omega)\| = \lambda_i = \lim_{n \to \pm\infty} \frac{1}{n} \ln \|x_n(\omega; \bar{x}(\omega))\|, \quad i = 1, \ldots, \tau,$$

*where* $\bar{x} = \Pi_{-1}\overline{\Pi}_0 x$.

## 6. Examples

**Example 6.1.** We consider LIDE

$$A_n x_{n+1} = B_n x_n + q_n, \tag{6.1}$$

where for $n \geqslant 0$,

$$A_n := \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_n := \begin{pmatrix} 1 & -1 & 1 \\ -n^2 & 0 & n \\ -n-1 & 1 & 0 \end{pmatrix}, \quad q_n := \begin{pmatrix} 1-2n \\ 0 \\ 2n \end{pmatrix}.$$

It is easy to see that $N_n = \ker A_n = \operatorname{span}\{(0,0,1)^\top\}$, $S_n = \operatorname{span}\{(1, n+1, 0)^\top, (0,0,1)^\top\}$ and

$$Q_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a projection onto $\ker A_n$ for all $n \geqslant 0$. Therefore, for any $n \geqslant 0$ we have that $N_{n-1} \cap S_n = \operatorname{span}\{(0,0,1)^\top\}$, i.e., $\dim(N_{n-1} \cap S_n) = 1, \ n \geqslant 0$.

Next, we can choose

$$T_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

to obtain

$$G_n := A_n + B_n T_n Q_n = \begin{pmatrix} 1 & 0 & 1 \\ n & 1 & n \\ 0 & 0 & 0 \end{pmatrix}, \quad N_{1,n} = \ker G_n = \operatorname{span}\{(-1, 0, 1)^\top\},$$

$$S_{1,n} = \operatorname{span}\{(1, n+1, 0)^\top, (0, 0, 1)^\top\}, \quad \text{and} \quad S_{1,n} \cap N_{1,n-1} = \{0\}, \quad n \geqslant 0.$$

Therefore, Eq. (6.1) is of index-2. Further,

$$\widehat{Q}_{1,n} = \begin{pmatrix} 1 & \frac{-1}{n+2} & 0 \\ 0 & 0 & 0 \\ -1 & \frac{1}{n+2} & 0 \end{pmatrix}$$

is a projection onto $\ker G_n$ along $S_{1,n+1}$. Let

$$T_{1,n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

simple calculation shows that

$$\widehat{G}_{1,n} = \begin{pmatrix} 2 & -\frac{1}{n+2} & 1 \\ n - n^2 & \frac{n^2+n+2}{n+2} & n \\ -n-1 & \frac{n+1}{n+2} & 0 \end{pmatrix}, \quad \Pi_{n-1} = \begin{pmatrix} 0 & \frac{1}{n+1} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{n}{n+1} & 0 \end{pmatrix}, \quad n \geqslant 1,$$

and

$$P_{-1}\widehat{P}_{1,-1} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi_{-1} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, by (4.25) in Theorem 4.7 we obtain

$$\begin{aligned} x_n = {} & \Pi_{n-1} u_n + T_n Q_n T_{1,n+1} \widehat{Q}_{1,n+1} \widehat{G}_{1,n+1}^{-1} q_{n+1} \\ & - T_n Q_n \widehat{P}_{1,n} \widehat{G}_{1,n}^{-1} q_n - P_{n-1} T_{1,n} \widehat{Q}_{1,n} \widehat{G}_{1,n}^{-1} q_n = \begin{pmatrix} 1 \\ 1-n \\ n \end{pmatrix}, \quad n \geqslant 1 \end{aligned}$$

and with $u_0 = P_{-1}\widehat{P}_{1,-1} x^0$, where $x^0 = (a, b, c)^\top \in \mathbb{R}^3$ we have

$$x_0 = \Pi_{-1} u_0 + T_0 Q_0 T_{1,1} \widehat{Q}_{1,1} \widehat{G}_{1,1}^{-1} q_1 - T_0 Q_0 \widehat{P}_{1,0} \widehat{G}_{1,0}^{-1} q_0 - P_{-1} T_{1,0} \widehat{Q}_{1,0} \widehat{G}_{1,0}^{-1} q_0 = \begin{pmatrix} \frac{b}{2} \\ b \\ 0 \end{pmatrix}.$$

**Example 6.2.** We consider a stochastic differential algebraic equation

$$AdX_t = BX_t dt + CX_t dW_t, \quad t \in \mathbb{R},\ X_0 \in \mathbb{R}^m, \tag{6.2}$$

where $A,\ B,\ C$ are $m \times m$-constant matrices with $\det(A) = 0$. In [11], author has investigated (6.2) with the assumption that the deterministic part, i.e., the equation $AdX_t = BX_t dt$, is index-1 tractable. However, as far as we know, the general conditions on the existence of the solution for Cauchy problem and how to solve (6.2) is still an open question. Using the explicit Euler method with stepsize $\tau$ we obtain a difference equation

$$AX_{(n+1)\tau} = (A + \tau B + \xi_n C)X_{n\tau}, \tag{6.3}$$

where $\xi_n = W_{(n+1)\tau} - W_{n\tau}$. Since $(\xi_n)$ is an i.i.d. sequence with the common distribution $N(0, \tau)$, there exists a preserving-measure transformation $\theta$ such that $\xi_n = \xi_0(\theta^n)$. Thus, the equation (6.3) is an example of (5.1).

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ -2 & 0 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

For the case $n \geqslant 0$ we choose

$$Q := Q_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to be a projection onto $\ker A_n$ and $T_n = I$. By direct calculation we have

$$G_n = A_n + B_n T_n Q_n = A + B_n Q = \begin{pmatrix} 1 & 0 & \tau + \xi_n \\ 1 & 1 & \tau + \xi_n \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear $\ker G_n = \operatorname{span}\{(\tau + \xi_n, 0, -1)^\top\}$. Thus, choosing

$$T_{1,n} = \operatorname{diag}\left(\frac{\tau + \xi_{n-1}}{\tau + \xi_n}, 1, 1\right)$$

and $Q_{1,n} = \begin{pmatrix} 0 & 0 & -(\tau + \xi_n) \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we see that Eq. (6.3) is index-2 tractable. Further,

$$\widehat{Q}_{1,n-1} = \begin{pmatrix} 1 & -\frac{1}{2}\frac{\xi_n}{\tau+\xi_n} & 0 \\ 0 & 0 & 0 \\ -\frac{1}{\tau+\xi_{n-1}} & \frac{1}{2}\frac{\xi_n}{(\tau+\xi_n)(\tau+\xi_{n-1})} & 0 \end{pmatrix},$$

and

$$\Pi_{n-1} = \begin{pmatrix} 0 & a_{n-1} & 0 \\ 0 & 1 & 0 \\ 0 & b_{n-1} & 0 \end{pmatrix}, \quad \Pi_n \widehat{G}_{1,n}^{-1} B_n = (\tau + \xi_n + 1) \begin{pmatrix} 0 & a_n & 0 \\ 0 & 1 & 0 \\ 0 & b_n & 0 \end{pmatrix}, \tag{6.4}$$

where

$$a_n = \frac{1}{2}\frac{\xi_{n+1}}{\tau + \xi_{n+1}}, \quad b_n = \frac{\tau}{2}\frac{(-\xi_{n+1}^2 + 3\tau\xi_{n+1} + 3\tau\xi_{n+2} + 2\tau^2 + 5\xi_{n+2}\xi_{n+1} - \xi_{n+1} + \xi_{n+2})}{(\tau + \xi_{n+1})^2(\tau + \xi_{n+2})}.$$

Hence, the cocycle $\Phi(n, \omega)$ is

$$\Phi(n, \omega) = \Pi_n \widehat{G}_{1,n}^{-1}(\omega) B_n(\omega) \prod_{i=0}^{n-1} (\tau + 1 + \xi_i(\omega)). \tag{6.5}$$

Since $(\xi_n)$ is an i.i.d. sequence with the common distribution $N(0, \tau)$ and taking into account (6.4), we get that $\Pi_n \widehat{G}_{1,n}^{-1}(\omega) B_n(\omega)$ is a stationary sequence with

$$\ln \|\Pi_n \widehat{G}_{1,n}^{-1}(\omega) B_n(\omega)\| \in L_1(\Omega, \mathcal{F}, P).$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \ln \|\Pi_n \widehat{G}_{1,n}^{-1} B_n\| = 0.$$

Applying the law of large numbers we have

$$\lambda_1 = \mathbb{E}\left(\ln|\tau + 1 + \xi_1|\right)$$

with

$$W_0 = \mathrm{span}\{(1,0,0)^\top, (0,0,1)^\top\},$$

and

$$W_1 = \mathrm{span}\{(0,1,0)^\top\}.$$

The backward equation, i.e., when $n < 0$, is of index-0. It is easy to obtain the formula for $\Phi(n,\omega)$,

$$\Phi(n) = \prod_{i=n}^{-1}(A + \tau B + \xi_i)^{-1} A\Pi_i = \prod_{i=n}^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{1+\tau+\xi_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \left(\prod_{i=n}^{-1}(1 + \tau + \xi_i)^{-1}\right)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, using the law of large number again, we obtain

$$\lim_{n\to-\infty}\frac{1}{n}\ln\|\Phi(n,\omega)x\| = \lambda_1 \quad \text{for any } 0 \neq x \in W_1.$$


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## Appendix A. Some surveys on linear algebra

In this section, we survey some basic properties of linear algebra. Let $(A, \overline{A}, B)$ be a triple of matrices. Suppose that $\mathrm{rank}\,A = \mathrm{rank}\,\overline{A} = r$ and $T \in GL(\mathbb{R}^m)$ is a transformation such that $T|_{\ker A}$ is an isomorphism from $\ker A$ onto $\ker \overline{A}$. We can give a such operator $T$ by the following way: let $Q$ (resp. $\overline{Q}$) be a projection onto $\ker A$ (resp. onto $\ker \overline{A}$); find nonsingular matrices $V$ and $\overline{V}$ such that $Q = VQ^{(0)}V^{-1}$ and $\overline{Q} = \overline{V}Q^{(0)}\overline{V}^{-1}$ where $Q^{(0)} = \mathrm{diag}(O, I_{m-r})$ with $I_k$ to be the $k \times k$-identity matrix; and finally we obtain $T$ by putting $T =: \overline{V}V^{-1}$.

Denote $S := \{x \in \mathbb{R}^m : Bx \in \mathrm{im}\,A\}$ and $P := I - Q$.

**Lemma A.1.** *The following assertions are equivalent*

(i) *the matrix $G := A + BTQ$ is nonsingular*;
(ii) $\mathbb{R}^m = \ker \overline{A} \oplus S$;
(iii) $S \cap \ker \overline{A} = \{0\}$.

**Proof.** (i) $\to$ (ii) It is obvious

$$x = (I - TQG^{-1}B)x + TQG^{-1}Bx =: x_1 + x_2$$

for any $x \in \mathbb{R}^m$. Since $Q$ is a projection onto $\ker A$ and $T|_{\ker A}$ is an isomorphism between $\ker A$ and $\ker \overline{A}$, it follows that $x_2$ belongs to $\ker \overline{A}$. Further,

$$Bx_1 = B(I - TQG^{-1}B)x = (B - BTQG^{-1}B)x = (B - (G - A)G^{-1}B)x$$
$$= (B - B + AG^{-1}B)x = AG^{-1}Bx \in \mathrm{im}\,A,$$

i.e., $x_1 \in S$. Thus, we obtain that $\mathbb{R}^m = \ker \overline{A} + S$. It remains to show that $\ker \overline{A} \cap S = \{0\}$. To this end, let $x \in S \cap \ker \overline{A}$, this means $x \in S$ and $x \in \ker \overline{A}$. Since $x \in S$, there exists $z \in \mathbb{R}^m$ such that

$Bx = Az = APz$ and from $x \in \ker \overline{A}$, we have $T^{-1}x \in \ker A$. Therefore, $T^{-1}x = QT^{-1}x$. This relation implies $(A + BTQ)T^{-1}x = (A + BTQ)Pz$. Hence, it follows that $T^{-1}x = Pz$. Thus, $T^{-1}x = 0$ and then $x = 0$. So, we come to the assertion (ii).

(ii) $\rightarrow$ (iii) This holds trivially by definition.

(iii) $\rightarrow$ (i) Let $x \in \mathbb{R}^m$ be a vector such that $Gx = 0$, i.e., $BTQx = -Ax$, This relation says that $TQx \in S$. On the other hand, $TQx$ lies in $\ker \overline{A}$. Using the fact $\ker \overline{A} \cap S = \{0\}$ we obtain $TQx = 0$, or $Qx = 0$ which yields $Ax = 0$. Therefore, $x \in \ker Q$ and $x \in \ker A$. It implies that $x = Qx = 0$. Hence, $G$ is nonsingular. The proof of Lemma A.1 is completed. □

**Lemma A2.** *Suppose that the matrix G is nonsingular. Then, there hold the following relations*:

(i) $$P = G^{-1}A; \tag{A.1}$$

(ii) $$G^{-1}BTQ = Q; \tag{A.2}$$

(iii) $\widetilde{Q} := TQG^{-1}B$ *is a projection onto* $\ker \overline{A}$ *along* $S$;

(iv) *if* $\overline{Q}$ *is a projection onto* $\ker \overline{A}$,

$$G^{-1}B = G^{-1}B\overline{P} + T^{-1}\overline{Q} \tag{A.3}$$

*with* $\overline{P} := I - \overline{Q}$.

**Proof**

(i) Noting $GP = (A + BTQ)P = AP = A$, we get (A.1).

(ii) From $BTQ = G - A$ and using the relation (A.1), we obtain $G^{-1}BTQ = I - P = Q$. Thus, we have (A.2).

(iii) By virtue of (A.2), $\widetilde{Q}^2 = TQG^{-1}BTQG^{-1}B = TQQG^{-1}B = TQG^{-1}B = \widetilde{Q}$. Further, since $T|_{\ker A}$ is an isomorphism from $\ker A$ onto $\ker \overline{A}$, it implies $\overline{A}TQG^{-1}B = 0$. Hence, $\widetilde{Q}$ is a projection onto $\ker \overline{A}$. On the other hand, from the proof of (iii) of Lemma A.1, it is the projection onto $\ker \overline{A}$ along $S$.

(iv) We have

$$G^{-1}B = G^{-1}B\overline{P} + G^{-1}B\overline{Q}.$$

Since $T^{-1}\overline{Q}x \in \ker A$ for any $x$, it yields that

$$G^{-1}B\overline{Q} = G^{-1}BTT^{-1}\overline{Q} = G^{-1}(A + BTQ)QT^{-1}\overline{Q} = QT^{-1}\overline{Q} = T^{-1}\overline{Q}.$$

Hence, we come to Eq. (A.3). Lemma A.2 is proved. □

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